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Geometric Aspects of the Riccati Difference Equation in the Nonsymmetric Case

Domenico D'Alessandro*
Dipartimento di Elettronica ed Informatica
Università di Padova
Via Gradenigo 6/A
351351 Padova, Italy

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ABSTRACT

This paper provides a collection of geometric results concerning the nonsymmetric Riccati difference equation. It gives representation formulas which express all the solutions in terms of a suitable number of known ones. Further results include geometric properties of the difference of two solutions and the generalization of some of the author's earlier results to the nonsymmetric case. © Elsevier Science Inc., 1997

1. INTRODUCTION

We define the nonsymmetric Riccati difference equation as

$$X(t+1) = [B_{21}(t) + B_{22}(t)X(t)][B_{11}(t) + B_{12}(t)X(t)]^{-1}, \quad (1.1)$$

where $X(t)$ is a complex matrix valued function of dimensions $m \times n$, on an interval of $\mathbb{Z}$, $[t_0, t_1]$, and $B_{ij}(t)$, $i, j = 1, 2$, are complex matrix valued functions, of suitable dimensions, in $[t_0, t_1 - 1]$.

*E-mail: daless@inco.dei.unipd.it.

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This equation has received much less attention in the past than its continuous time counterpart, the Riccati differential equation (see [12, 18, 10, 13]). This is probably due to the fact that the latter equation is also of independent mathematical interest, since it describes, in local coordinates, a linear differential equation on the Grassmann manifold. In the discrete time context, a local description is not significant. However, we shall see that also Equation (1.1) has an immediate interpretation on the Grassmann manifold [see (3.1), (3.3) below]. Equation (1.1) has received less attention even than the symmetric Riccati difference equation (see [19, 9, 1, 4, 22]), whose importance in systems and control theory is well known. Nevertheless, there are many cases of practical interest in which the Riccati difference equation, in the nonsymmetric form (1.1), occurs.

A remarkable example is given by the following equation, whose solutions determine optimal strategies in discrete-time LQ dynamic games problems [2]:

$$X(t+1) = Q(t) + A^*(t)X(t)[I + L(t)X(t)]^{-1}A(t).$$

If we suppose $A(t)$ nonsingular in the interval $[t_0, t_1 - 1]$, this equation can be obtained from (1.1) by setting, in $[t_0, t_1 - 1]$, $B_{11}(t) = A^{-1}(t)$, $B_{12}(t) = A^{-1}(t)L(t)$, $B_{21}(t) = Q(t)A^{-1}(t)$, and $B_{22}(t) = Q(t)A^{-1}(t)L(t) + A^*(t)$. The previous equation also appears in disturbance rejection problems, which are related to the dynamic games setting [3]. In these contexts, solutions of nonsymmetric Riccati difference equations of the type (1.1) also determine the existence of optimal strategies.

A further example of occurrence of (1.1) is the projection preserving difference equation

$$X(t+1) = H(t)[I + X(t)L(t)]^{-1}X(t)M^{-1}(t),$$

where $H(t)$, $L(t)$, $M(t)$ are $n \times n$ matrix valued functions in $[t_0, t_1 - 1]$, the last being nonsingular, and such that

$$M(t)[I + L(t)] = H(t), \qquad t \in [t_0, t_1 - 1].$$

In this case we have $B_{11}(t) = M(t)$, $B_{12}(t) = M(t)L(t)$, $B_{21}(t) = 0$ and $B_{22}(t) = M(t)[I + L(t)]$. The Riccati difference equation of the least squares optimal control problem,

$$\begin{aligned} X(t+1) = {} & F^*(t)X(t)F(t) - F^*(t)X(t)G(t) \\ & \times [I + G^*(t)X(t)G(t)]^{-1}G^*(t)X(t)F(t) + Q(t), \end{aligned} \tag{1.2}$$

is a special case of (1.1), when $F(t)$ is nonsingular, if we set in $[t_0, t_1 - 1]$, $B_{11}(t) = F^{-1}(t)$, $B_{12}(t) = F^{-1}(t)G(t)G^*(t)$, $B_{21}(t) = Q(t)F^{-1}(t)$, and $B_{22}(t) = F^*(t) + Q(t)F^{-1}(t)G(t)G^*(t)$.

In this paper, a study of the geometry of Equation (1.1) is presented. The main results are in Section 3 and Section 4 below. In particular, in Section 3, we give two representation formulas for the computation of solutions of (1.1), when some of them are known. One formula allows us to calculate all the solutions of (1.1) in terms of l $[m \leqslant (l-1)n]$ of them. The other one identifies families of solutions which are projective superpositions of known solutions, thus extending to the nonsymmetric case the main result of [6]. Interest in such formulas is motivated, for instance, in linear quadratic optimal control or in dynamic games problems, when the cost index presents a penalty term. In these cases, the optimal strategy is typically a function of the solution of (1.1) whose boundary value is determined by the penalty term. It is of interest to vary this term with respect to previous situations, and having a representation formula such as the ones proved here allows us to avoid integrating the equation every time. Only certain solutions have to be calculated. The data obtained in this way can be used as a data bank for the computation of other solutions, and this results in computational savings (similar problems are dealt with in [20] for the Riccati differential equation). Also, formulas such as the ones proved here are very useful when two boundary value problems are considered, since they allow to relate, in a simple manner, the values of a solution at two different points.

In Section 4, we turn to the study of the comparative properties of solutions of Equation (1.1), namely geometric properties of the differences of two solutions. We start by extending to the nonsymmetric case a result of [8] where a homogeneous equation is given for these differences, and conclude by stating properties of the equation concerning their ranges and null spaces.

2. DEFINITIONS AND BASIC ASSUMPTIONS

We associate to Equation (1.1) the following complex matrix valued function of dimensions $(m+n) \times (m+n)$:

$$B(t) \doteq \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix}, \qquad t \in [t_0, t_1 - 1]. \tag{2.1}$$

In the sequel, we shall assume that this matrix is invertible. An important special case in which this assumption is verified, is for Equation (1.2), when

$F(t)$ is nonsingular. In this case, as a simple verification shows, $B(t)$ has the property of being J-unitary, i.e. $B^*(t)JB(t) = J$, $t \in [t_0, t_1 - 1]$, where J is defined as

$$J \doteq \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

Therefore, $B(t)$, $t \in [t_0, t_1 - 1]$, is invertible, and we have

$$B^{-1}(t) = -JB^*(t)J, \qquad t \in [t_0, t_1 - 1].$$

We denote the inverse of (2.1) by

$$B^{-1}(t) \doteq \begin{pmatrix} B_{11}^-(t) & B_{12}^-(t) \\ B_{21}^-(t) & B_{22}^-(t) \end{pmatrix}, \qquad t \in [t_0, t_1 - 1], \tag{2.2}$$

where the partition is consistent with the one in (2.1).

The following result gives a very general property of a partitioned square matrix and of its inverse. It is, to the best of our knowledge, a new result in matrix analysis, and it will allow us to state some definitions.

LEMMA 2.1. *Consider an invertible matrix B, and partition it as in (2.1), i.e., perform on it an arbitrary partition into four blocks, with the only requirement that the diagonal blocks are square. Then define the corresponding partition on its inverse, as in (2.2). For each matrix X of suitable dimensions,*

$$\det(B_{11} + B_{12}X) = \det B \det(B_{22}^- - XB_{12}^-). \tag{2.3}$$

Proof. First notice that

$$\det B = \det \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \begin{pmatrix} I & 0 \\ X & I \end{pmatrix}$$

$$= \det \begin{pmatrix} B_{11} + B_{12}X & B_{12} \\ B_{21} + B_{22}X & B_{22} \end{pmatrix}.$$

From this we obtain

$$\det B \det(B_{22}^- - XB_{12}^-) = \det\begin{pmatrix} B_{12} + B_{12}X & B_{12} \\ B_{21} + B_{22}X & B_{22} \end{pmatrix}\begin{pmatrix} I & B_{12}^- \\ 0 & B_{22}^- - XB_{12}^- \end{pmatrix}$$

$$= \det\begin{pmatrix} B_{11} + B_{12}X & B_{11}B_{12}^- + B_{12}B_{22}^- \\ B_{21} + B_{22}X & B_{21}B_{12}^- + B_{22}B_{22}^- \end{pmatrix}$$

$$= \det\begin{pmatrix} B_{11} + B_{12}X & 0 \\ B_{21} + B_{22}X & I \end{pmatrix} = \det(B_{11} + B_{12}X). \quad \blacksquare$$

We shall refer to a complex matrix valued function $X(t)$ as a solution of (1.1) in $[t_0, t_1]$ if it satisfies (1.1) with $B_{11}(t) + B_{12}(t)X(t)$ nonsingular in $[t_0, t_1 - 1]$, so that the inversion in (1.1) can be performed. Given such a solution, we define a *forward* feedback matrix as the $n \times n$ matrix

$$F_X^+(t) \doteq [B_{11}(t) + B_{12}(t)X(t)]^{-1}, \qquad t \in [t_0, t_1 - 1], \tag{2.4}$$

and a *backward* feedback matrix (using Lemma 2.1) as the $m \times m$ matrix

$$F_X^-(t) \doteq [B_{22}^-(t) - X(t)B_{12}^-(t)]^{-1}, \qquad t \in [t_0, t_1 - 1]. \tag{2.5}$$

These definitions generalize known ones for Equation (1.2). In fact, in this special case, using the identity $B^{-1}(t) = -JB^*(t)J$, we can calculate the expression for the inverse matrix of $B(t)$, and show that

$$F_X^+(t) = (F_X^-)^*(t) = F(t) - G(t)[I + G^*(t)X(t)G(t)]^{-1}G^*(t)X(t)F(t)$$

$$= [I + G(t)G^*(t)X(t)]^{-1}F(t).$$

This is the usual feedback matrix defined in systems theory and least squares optimal control problems (see e.g. [1]).

We recall next, from systems theory, the definition of a transition matrix $\Psi(t, s)$ associated to a given matrix $A(t)$, i.e.

$$\begin{aligned} &\Psi(t, s) \doteq A(t-1)\cdots A(s), \qquad t_0 \leqslant s < t \leqslant t_1, \\ &\Psi(s, s) = I, \qquad t_0 \leqslant s \leqslant t_1. \end{aligned} \tag{2.6a}$$

Moreover, if $A(t)$ is nonsingular in $[t_0, t_1 - 1]$, then $\Psi(t, s)$ is invertible and we can define it even for values $t < s$, $t, s \in [t_0, t_1]$, as

$$\Psi(t, s) = [\Psi(s, t)]^{-1}, \qquad t_0 \leqslant t < s \leqslant t_1. \tag{2.6b}$$

In the sequel, we shall specialize the previous definitions in various ways. In particular, we shall be concerned with transition matrices related to the feedback matrices defined above. These, under our assumptions, are invertible.

3. REPRESENTATION FORMULAS

In order to state representation formulas for the computation of solutions of Equation (1.1), we need to recall some notions from the geometric approach to the study of the Riccati equation. These are presented in the literature (see [14, 18, 10]) for the Riccati differential equation; however, they can be easily adapted to the context of the Riccati difference equation considered here.

Consider the linear matrix system

$$\begin{pmatrix} U(t+1) \\ V(t+1) \end{pmatrix} = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} \begin{pmatrix} U(t) \\ V(t) \end{pmatrix} \tag{3.1}$$

in the interval $[t_0, t_1]$. Here $V(t)$, $U(t)$ are respectively $m \times n$ and $n \times n$ complex matrix valued functions in $[t_0, t_1]$. It is easily seen that, if we consider as initial value

$$\begin{pmatrix} U(t_0) \\ V(t_0) \end{pmatrix} = \begin{pmatrix} I_n \\ X(t_0) \end{pmatrix}, \tag{3.2}$$

in (3.1), then

$$X(t) = V(t)U(t)^{-1} \tag{3.3}$$

gives the solution of (1.1) corresponding to the initial value $X(t_0)$, as long as the indicated inverse exists. Furthermore, the occurrence of the singularity of the matrix $U(t+1)$ corresponds to the impossibility of performing the inversion in (1.1) at t. If we replace the matrices $U(t)$, $V(t)$ in (3.1) by

$U(t)R$, $V(t)R$ respectively, where R is an arbitrary nonsingular matrix of dimensions $n \times n$, we obtain the same solution. Therefore, we can look at Equation (3.1) as a linear equation on the Grassmann manifold of the n-dimensional subspaces of C^{n+m}. There is a one to one correspondence between the time varying subspaces spanned by $\binom{U(t)}{V(t)}$, and complementary to span$\binom{0}{I_m}$, [i.e. such that $U(t)$ is invertible], and the solutions of (1.1). This correspondence is given by (3.3).

The previous discussion suggests a two step strategy in the study of the solutions of (1.1): First examine the system (3.1) on the interval $[t_0, t_1]$, and then recover the behavior of the corresponding solution of (1.1) using (3.3). The latter equation gives the discrete time counterpart of Radon's formula [14], and the above-described phenomenon of having $U(t)$ singular corresponds to the occurrence of a finite escape time in the case of the differential equation [11, 5].

The next theorem gives a representation formula for the computation of solutions of the Riccati difference equation (1.1). In its proof, we shall assume that $m \leqslant n$. In this case two solutions, chosen in an appropriate way, are sufficient to describe all the solutions of (1.1). The extension to the general case uses the same idea and involves more complex notations. It will be sketched in the following remarks.

THEOREM 3.1. *Consider two solutions $X_1(t)$, $X_2(t)$ of (1.1) such that $\Delta_{21}(t_0) \doteq X_2(t_0) - X_1(t_0)$ admits a right inverse, which we denote by $\Delta_{21}^{-R}(t_0)$. Also define $\Delta_k(t_0) \doteq X(t_0) - X_k(t_0)$, $k = 1, 2$. Then, consider the $n \times n$ matrix functions, $U_k(t)$, $k = 1, 2$, $t \in [t_0, t_1]$, which satisfy*

$$U_k(t+1) = \left[F_{X_k}^{+}(t)\right]^{-1} U_k(t), \quad t \in [t_0, t_1], \qquad U_k(t_0) = I_n. \tag{3.4}$$

Then, each solution of (1.1) is given by

$$X(t) = \left\{X_j(t)U_j(t) + \left[X_2(t)U_2(t) - X_1(t)U_1(t)\right]\Delta_{21}^{-R}(t_0)\Delta_j(t_0)\right\}$$

$$\times\left\{U_i(t) + \left[U_2(t) - U_1(t)\right]\Delta_{21}^{-R}(t_0)\Delta_i(t_0)\right\}^{-1}, \tag{3.5}$$

for an arbitrary pair $i, j = 1, 2$.

Proof. For any solution $X(t)$ of (1.1), using (3.1)–(3.3), we have, for $t \in [t_0, t_1]$,

$$\begin{pmatrix} U(t+1) \\ X(t+1)U(t+1) \end{pmatrix} = \begin{pmatrix} B_{11}(t) & B_{12}(t) \\ B_{21}(t) & B_{22}(t) \end{pmatrix} \begin{pmatrix} U(t) \\ X(t)U(t) \end{pmatrix}, \qquad U(t_0) = I_n. \tag{3.6}$$

In particular, for $X_1(t)$, $X_2(t)$, the function $U(t)$ satisfies (3.4). Accordingly to the partition of $B(t)$, we define a partition on its transition matrix as

$$\Psi_B(t,s) \doteq \begin{pmatrix} \Psi_B^{11}(t,s) & \Psi_B^{12}(t,s) \\ \Psi_B^{21}(t,s) & \Psi_B^{22}(t,s) \end{pmatrix}, \qquad s,t \in [t_0, t_1].$$

With this definition, we write (3.6) as

$$\begin{pmatrix} U(t) \\ X(t)U(t) \end{pmatrix} = \begin{pmatrix} \Psi_B^{11}(t,t_0) & \Psi_B^{12}(t,t_0) \\ \Psi_B^{21}(t,t_0) & \Psi_B^{22}(t,t_0) \end{pmatrix} \begin{pmatrix} I_n \\ X(t_0) \end{pmatrix}, \qquad t \in [t_0, t_1]. \tag{3.7}$$

Therefore, any solution of (1.1) can be expressed as

$$X(t) = \left[\Psi_B^{21}(t,t_0) + \Psi_B^{22}(t,t_0)X(t_0)\right]\left[\Psi_B^{11}(t,t_0) + \Psi_B^{12}(t,t_0)X(t_0)\right]^{-1}. \tag{3.8}$$

In our situation, the transition matrix can be completely reconstructed by the knowledge of two solutions. $U_k(t)$, $k = 1, 2$, can be obtained by (3.4). Specializing (3.7), we have

$$U_k(t) = \left[\Psi_B^{11}(t,t_0) + \Psi_B^{12}(t,t_0)X_k(t_0)\right], \quad k = 1,2, \ t \in [t_0, t_1],$$

$$X_k(t)U_k(t) = \left[\Psi_B^{21}(t,t_0) + \Psi_B^{22}(t,t_0)X_k(t_0)\right], \quad k = 1,2, \ t \in [t_0, t_1].$$

We obtain, for $t \in [t_0, t_1]$,

$$
\begin{aligned}
\Psi_B^{12}(t,t_0) &= [U_2(t) - U_1(t)]\Delta_{21}^{-R}(t_0),\\
\Psi_B^{11}(t,t_0) &= U_1(t) - [U_2(t) - U_1(t)]\Delta_{21}^{-R}(t_0)X_1(t_0)\\
&= U_2(t) - [U_2(t) - U_1(t)]\Delta_{21}^{-R}(t_0)X_2(t_0),\\
\Psi_B^{22}(t,t_0) &= [X_2(t)U_2(t) - X_1(t)U_1(t)]\Delta_{21}^{-R}(t_0),\\
\Psi_B^{21}(t,t_0) &= X_1(t)U_1(t) - [X_2(t)U_2(t) - X_1(t)U_1(t)]\Delta_{21}^{-R}(t_0)X_1(t_0)\\
&= X_2(t)U_2(t) - [X_2(t)U_2(t) - X_1(t)U_1(t)]\Delta_{21}^{-R}(t_0)X_2(t_0).
\end{aligned}
\tag{3.9}
$$

Plugging these into (3.8), and recalling the definition of $\Delta_k(t_0)$, we get (3.5). ■

REMARKS.

(i) The only hypothesis under which the above result is established is that the matrix $B(t)$ is nonsingular in $[t_0, t_1 - 1]$. Moreover, for the practical computation of solutions, the theorem requires that the reference solutions $X_1(t)$, $X_2(t)$ be selected so that $\Delta_{21}(t_0) \doteq X_2(t_0) - X_1(t_0)$ admits a right inverse. This is always possible. The assumption that $m \leqslant n$ is just for notational convenience. In fact, the theorem exploits the idea of reconstructing a curve in the group $SL(m + n, C)$, namely $\Psi_B(t, t_0)$, using known solutions. When $m > n$, more than two solutions have to be used. In particular, the minimum number of them is the minimum l such that $m \leqslant (l - 1)n$. In this case, it is possible to choose the reference solutions $X_1(t), \dots, X_l(t)$ so that, with $\Delta_{lj}(t_0) \doteq X_l(t_0) - X_j(t_0)$, $j = 1, \dots, l - 1$, the $m \times (l - 1)n$ matrix $(\Delta_{l1}(t_0) \;\cdots\; \Delta_{l(l-1)}(t_0))$ admits a right inverse. With this choice, and having l equations for $U(t)$ and $X(t)U(t)$, the generalization for $m > n$ is a simple adaptation of the above proof.

(ii) In [20] formulas such as the one proved in Theorem 3.1 are called *2-representation formulas*, since they involve the knowledge of two solutions and the integration of a of a set of auxiliary linear (difference) equations. In that paper, the Riccati differential equation is dealt with. The above result can be also readily derived in that setting, since it is essentially based on the geometric approach, described at the beginning of this section, which applies to both the discrete and the continuous time case. In fact, our result represents an improvement with respect to the 2-representation formula presented in [20] in that it does not require the equation to be square (i.e. $m = n$) and applies to all the solutions $X(t)$, while the formula given in [20] only applies when $\Delta_1(t_0) \doteq X(t_0) - X_1(t_0)$ is nonsingular.

(iii) A particularly interesting case is for the time invariant Riccati difference equation when the reference solutions $X_1(t)$, $X_2(t)$ can be chosen as equilibrium solutions. In this case, the solutions of (3.4) can be given a simple expression, $U_k(t) = (F_{X_k}^+)^{-t}$, $t \in [t_0, t_1]$, $k = 1, 2$, and (3.5) reads as

$$X(t) = \left\{X_j\left(F_{X_j}^+\right)^{-t} + \left[X_2\left(F_{X_2}^+\right)^{-t} - X_1\left(F_{X_1}^+\right)^{-t}\right]\Delta_{21}^{-R}(t_0)\Delta_j(t_0)\right\}$$
$$\times\left\{\left(F_{X_i}^+\right)^{-t} + \left[\left(F_{X_2}^+\right)^{-t} - \left(F_{X_1}^+\right)^{-t}\right]\Delta_{21}^{-R}(t_0)\Delta_i(t_0)\right\}^{-1},$$
$$i, j = 1, 2.$$

For the Riccati difference equation of optimal control and estimation (1.2), classification results for the equilibrium solutions are available (see [15, 22, 16]).

(iv) In [6], we have been interested in giving the expression for solutions of the symmetric equation (1.2) in terms of projective combinations of particular solutions. More specifically, we considered two subspaces $M(t_0)$, $N(t_0)$ such that $M(t_0) \oplus N(t_0) = C^n$, and defined $M(t) = \overline{\Psi}_1(t, t_0)M(t_0)$, $N(t) = \overline{\Psi}_2(t, t_0)N(t_0)$, where $\overline{\Psi}_1(t, t_0)$ and $\overline{\Psi}_2(t, t_0)$ are the transition matrices relative to $(F_{X_1}^+)^{-1}(t)$ and $(F_{X_2}^+)^{-1}(t)$ respectively. We showed that, if $\Pi(t)$ projects timewise onto $M(t)$ along $N(t)$, then $X(t) = X_1(t)\Pi(t) + X_2(t)[I - \Pi(t)]$ satisfies (1.1). The generalization to the nonsymmetric case can be easily obtained from (3.5) when we consider the square equation and a pair $X_2(t)$, $X_1(t)$ such that $\Delta_{21}(t_0)$ is nonsingular. To this aim, particularize (3.5) (written for $i = 1$, $j = 1$) for an initial value $X(t_0) = X_1(t_0)\Pi(t_0) + X_2(t_0)[I - \Pi(t_0)]$. It is $\Delta_1(t_0) = \Delta_{21}(t_0)[I - \Pi(t_0)]$ and, for $t \in [t_0, t_1]$,

$$\begin{aligned} X(t) &= \{X_1(t)U_1(t) + [X_2(t)U_2(t) - X_1(t)U_1(t)]\Delta_{21}^{-1}(t_0)\Delta_1(t_0)\} \\ &\quad \times\{U_1(t) + [U_2(t) - U_1(t)]\Delta_{21}^{-1}(t_0)\Delta_1(t_0)\}^{-1} \\ &= \{X_1(t)U_1(t) + [X_2(t)U_2(t) - X_1(t)U_1(t)][I - \Pi(t_0)]\} \\ &\quad \times\{U_1(t) + [U_2(t) - U_1(t)][I - \Pi(t_0)]\}^{-1} \\ &= X_1(t)U_1(t)\Pi(t_0)\{U_1(t)\Pi(t_0) + U_2(t)[I - \Pi(t_0)]\}^{-1} \\ &\quad + X_2(t)\left(I - U_1(t)\Pi(t_0)\{U_1(t)\Pi(t_0) + U_2(t)[I - \Pi(t_0)]\}^{-1}\right). \end{aligned}$$

It is straightforward to verify that $U_1(t)\Pi(t_0)\{U_1(t)\Pi(t_0) + U_2(t)[I - \Pi(t_0)]\}^{-1}$ is a projection valued matrix for $t \in [t_0, t_1]$, and in particular, by (3.4), it is the above-defined matrix $\Pi(t)$.

The geometric property of the equation (1.1), of preserving projective combinations of solutions, is described in different settings in the above remark and in [6]. It is proven in full generality in the following theorem.

THEOREM 3.2. *Consider the Riccati difference equation* (1.1) *and its l solutions in $[t_0, t_1]$, say $X_1(t), \dots, X_l(t)$, with $l \geqslant 2$. Consider the forward feedback matrix associated to the solution $X_i(t)$, $i = 1, \dots, l$, i.e.*

$$F^+_{X_i}(t) \doteq \left[B_{11}(t) + B_{12}(t)X_i(t)\right]^{-1}, \qquad t \in [t_0, t_1 - 1],$$

and the corresponding transition matrices, $\overline{\Psi}_i(t, t_0)$, $i = 1, \dots, l$. Then select l subspaces $M_i(t_0)$, $i = 1, \dots, l$, such that

$$M_1(t_0) \oplus M_2(t_0) \oplus \cdots \oplus M_l(t_0) = C^n.$$

Define

$$M_i(t) = \overline{\Psi}_i(t, t_0) M_i(t_0), \qquad i = 1, \dots, l, \quad t \in [t_0, t_1]. \quad (3.10)$$

Define $X(t) \doteq \sum_{i=1}^{l} X_i(t)\Pi_i(t)$, where $\Pi_i(t)$ is timewise the projection onto the subspace $M_i(t)$ along the subspace $M_1(t) \oplus \cdots \oplus M_{i-1}(t) \oplus M_{i+1}(t) \oplus \cdots \oplus M_l(t)$, $i = 1, \dots, l$. Then $X(t)$ satisfies (1.1).

The proof of this theorem is based on the previously described geometric approach. It follows closely the one given in [7] for the continuous time case, and it is therefore deferred to the Appendix.

REMARKS.

(i) The above result gives a general rule for constructing families of solutions of (1.1) when $l \geqslant 2$ of them are known. The only hypothesis is that $B(t)$ is nonsingular in $[t_0, t_1 - 1]$. No restriction is placed on the reference solutions $X_1(t), \dots, X_l(t)$. This result generalizes the above-recalled result of [6] not only in that it pertains to the general nonsymmetric Riccati difference equation, but also in that it considers projective combinations of more than two reference solutions.

(ii) Representation formulas based on projective superposition laws, such as the above, have been proposed in various contexts in the study of Riccati equations. The first result in this direction has been the one of J. C. Willems [21], concerning the parametrization of solutions of the continuous time algebraic Riccati equation. This result has been extended in various manners for solutions of the Riccati differential equation [18, 13, 7]. Projective superposition laws have been considered much more recently for the discrete time equation in the symmetric case [6] (see also [22, 16] for the discrete time algebraic case).

The following proposition gives a necessary and sufficient condition for the existence of the above-defined families of solutions. Its proof is a straightforward consequence of the geometric construction at the beginning of this section and of the proof of Theorem 3.2.

PROPOSITION 3.3. *Let $X(t)$ be a solution of* (1.1) *constructed as in Theorem* 3.2. *Let $\overline{U}_i(t_0)$ be a basis of $M_i(t_0)$ for $i = 1, \dots, l$. Then $X(t)$ is such that* $\det[B_{11}(t) + B_{12}(t)X(t)] \neq 0$, $t \in [t_0, t_1 - 1]$, *if and only if the $n \times n$ matrix*

$$\left(\overline{\Psi}_1(t, t_0)\overline{U}_1(t_0)\right| \cdots \left|\overline{\Psi}_l(t, t_0)\overline{U}_l(t_0)\right)$$

is nonsingular for each $t \in [t_0, t_1]$

4. PROPERTIES OF DIFFERENCES

A standard tool in the analysis of the Riccati differential equation consists in relating its solutions to the ones of an associated homogeneous Riccati equation. This must be satisfied by the difference of two solutions of the original equation. This approach has been used for example in [17], for the analysis of solutions of the continuous time algebraic Riccati equation. The corresponding equation for (1.2) has been proposed only recently in [8]. Its proof entails much more involved calculations than for the continuous time case. Our starting point, in this section, is the generalization of this result to the nonsymmetric case.

THEOREM 4.1. *Consider two arbitrary solutions of* (1.1), *$X_1(t)$ and $X_2(t)$, in $[t_0, t_1]$. Define $\Delta_{21}(t) \doteq X_2(t) - X_1(t)$. Then the following relation holds*:

$$\Delta_{21}(t + 1) = F_{X_1}^-(t)\Delta_{21}(t)F_{X_2}^+(t) = F_{X_2}^-(t)\Delta_{21}(t)F_{X_1}^+(t). \quad (4.1)$$

Proof. The second equality follows immediately from the first one, by changing the roles of $X_1(t)$ and $X_2(t)$. Therefore, we only have to prove the first one. In order to do this, subtract from (1.1), written for $X_2(t)$, (1.1) written for $X_1(t)$. We get (omitting the argument whenever it is t)

$$\begin{aligned}
&\Delta_{21}(t+1)(B_{11}+B_{12}X_2)\\
&\quad = B_{21}+B_{22}X_2-(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}(B_{11}+B_{12}X_2)\\
&\quad = B_{22}(X_2-X_1)+(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}(B_{11}+B_{12}X_1)\\
&\qquad -(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}(B_{11}+B_{12}X_2)\\
&\quad = \left[B_{22}-(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}B_{12}\right](X_2-X_1). \qquad (4.2)
\end{aligned}$$

Now we claim that

$$B_{22}-(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}B_{12}=(B_{22}^{-}-X_1B_{12}^{-})^{-1}. \qquad (4.3)$$

In order to see this, calculate

$$\begin{aligned}
&\left[B_{22}-(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}B_{12}\right](B_{22}^{-}-X_1B_{12}^{-})\\
&\quad = B_{22}B_{22}^{-}-B_{22}X_1B_{12}^{-}-(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}B_{12}B_{22}^{-}\\
&\qquad +(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}B_{12}X_1B_{12}^{-}.
\end{aligned}$$

Using the fact that $B_{11}B_{12}^{-}+B_{12}B_{22}^{-}=0$, the latter expression is equal to

$$\begin{aligned}
&B_{22}B_{22}^{-}-B_{22}X_1B_{12}^{-}+(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}B_{11}B_{12}^{-}\\
&\qquad +(B_{21}+B_{22}X_1)(B_{11}+B_{12}X_1)^{-1}B_{12}X_1B_{12}^{-}\\
&\qquad = B_{22}B_{22}^{-}-B_{22}X_1B_{12}^{-}+(B_{21}+B_{22}X_1)B_{12}^{-}=I,
\end{aligned}$$

where we have used the fact that $B_{22}B_{22}^{-}+B_{21}B_{12}^{-}=I$. Plugging (4.3) into (4.2), we get (4.1) ■

The previous proposition also gives the counterpart, in our context, of the similarity relations between feedback matrices, known in the *algebraic case* (see e.g. [21]). We give below a more general interpretation of the previous result.

PROPOSITION 4.2. *With the definitions of Theorem* 4.1, *we have, for each* $s, t \in [t_0, t_1]$,

$$\Delta_{21}(t) = \Psi_1^-(t,s)\Delta_{21}(s)\big(\tilde{\Psi}_2\big)^*(t,s) = \Psi_2^-(t,s)\Delta_{21}(s)\big(\tilde{\Psi}_1\big)^*(t,s). \tag{4.4}$$

Here we have denoted by $\tilde{\Psi}_i(t, s)$, $i = 1, 2$, the transition matrix associated to $(F_{X_i}^+)^*(t)$, and by $\Psi_i^-(t, s)$, $i = 1, 2$, the one associated to $F_{X_i}^-(t)$.

Proof. We rewrite Equation (4.1) with $t = s$. We get

$$\Delta_{21}(s+1) = F_{X_1}^-(s)\Delta_{21}(s)F_{X_2}^+(s) = F_{X_2}^-(s)\Delta_{21}(s)F_{X_1}^+(s).$$

Applying this repeatedly, we get

$$\begin{aligned}\Delta_{21}(t) &= F_{X_1}^-(t-1)\cdots F_{X_1}^-(s)\Delta_{21}(s)F_{X_2}^+(s)\cdots F_{X_2}^+(t-1)\\ &= F_{X_2}^-(t-1)\cdots F_{X_2}^-(s)\Delta_{21}(s)F_{X_1}^+(s)\cdots F_{X_1}^+(t-1).\end{aligned}$$

Using the definition of a transition matrix (2.6), this is equivalent to (4.4). ■

From Proplosition 4.2, a series of results concerning the geometry of the matrix $\Delta_{21}(t)$, with t varying in $[t_0, t_1]$, can be derived.

COROLLARY 4.3. *The difference of two solutions of* (1.1), $\Delta_{21}(t)$, *has constant rank, in* $[t_0, t_1]$.

COROLLARY 4.4. *Consider three arbitrary solutions of* (1.1), $X_1(t)$, $X_2(t)$, $X(t)$, *in* $[t_0, t_1]$. *Define* $\Delta_1(t) \doteq X(t) - X_1(t)$. *If* $\operatorname{Ker}\Delta_{21}(s) \subseteq \operatorname{Ker}\Delta_1(s)$ *for one* $s \in [t_0, t_1]$, *then* $\operatorname{Ker}\Delta_{21}(t) \subseteq \operatorname{Ker}\Delta_1(t)$, *for each* $t \in [t_0, t_1]$.

Proof. Consider x such that $\Delta_{21}(t)x = 0$. From (4.4) it follows that

$$\Delta_{21}(t)x = \Psi_2^-(t,s)\Delta_{21}(s)\big(\tilde{\Psi}_1\big)^*(t,s)x = 0.$$

Using the invertibility of $\Psi_2^-(t,s)$, we have

$$\big(\tilde{\Psi}_1\big)^*(t,s)x \in \operatorname{Ker}\Delta_{21}(s).$$

By the assumption $\operatorname{Ker}\Delta_{21}(s) \subseteq \operatorname{Ker}\Delta_1(s)$, recalling (4.4), and denoting by $\Psi^-(t,s)$ the transition matrix associated to $F_X^-(t)$, we have

$$\Delta_1(t)x = \Psi^-(t,s)\Delta_1(s)\big(\tilde{\Psi}_1\big)^*(t,s)x = 0. \qquad \blacksquare$$

Dually we have

COROLLARY 4.5. *With the same definitions of Corollary* 4.4, *if* $\operatorname{Im}\Delta_{21}(s) \subseteq \operatorname{Im}\Delta_1(s)$ *for one* $s \in [t_0, t_1]$, *then* $\operatorname{Im}\Delta_{21}(t) \subseteq \operatorname{Im}\Delta_1(t)$, $t \in [t_0, t_1]$.

APPENDIX

Proof of Theorem 3.2. As discussed in Section 3, we set the following correspondence between the l solutions $X_1(t), \dots, X_l(t)$, in $[t_0, t_1]$, and l n-dimensional time varying subspaces, complementary to $\operatorname{span}\begin{pmatrix} 0 \\ I_m \end{pmatrix}$,

$$X_i(t) \Leftrightarrow \operatorname{span}\begin{pmatrix} U_i(t) \\ V_i(t) \end{pmatrix}, \qquad i = 1, \dots, l, \quad t \in [t_0, t_1].$$

Here $V_i(t)$, $U_i(t)$, $i = 1, \dots, l$ are respectively $m \times n$ and $n \times n$ matrix valued functions defined in $[t_0, t_1]$, with $U_i(t)$, $i = 1, \dots, l$, timewise nonsingular. We consider l disjoint subspaces in C^{m+m},

$$\operatorname{span}\begin{pmatrix} \overline{U}_i(t_0) \\ \overline{V}_i(t_0) \end{pmatrix},$$

of dimensions n_i, $i = 1, \dots, l$, $n_1 + n_2 + \cdots + n_l = n$, complementary to $\binom{0}{I_m}$, and such that

$$\operatorname{span}\begin{pmatrix} \bar{U}_i(t_0) \\ \bar{V}_i(t_0) \end{pmatrix} \subseteq \operatorname{span}\begin{pmatrix} U_i(t_0) \\ V_i(t_0) \end{pmatrix}, \qquad i = 1, \dots, l.$$

We can express these subspaces as

$$\operatorname{span}\begin{pmatrix} \bar{U}_i(t) \\ X_i(t)\bar{U}_i(t) \end{pmatrix}, \qquad i = 1, \dots, l, \quad t \in [t_0, t_1].$$

where $\bar{U}_i(t)$, $i = 1, \dots, l$, $t \in [t_0, t_1]$, is a $n \times n_i$ matrix of full rank. The n_i columns of $\bar{U}_i(t_0)$ constitute a basis for an n_i-dimensional subspace $M_i(t_0)$ in C^n. We require these subspaces to propagate as in (3.10). Theorefore, we also have

$$\bar{U}_i(t) = \bar{\Psi}_i(t, t_0)\bar{U}_i(t_0), \qquad i = 1, \dots, l, \quad t \in [t_0, t_1]. \tag{A.1}$$

It is an easy verification to show that

$$\begin{pmatrix} \bar{U}_1(t) & \cdots & \bar{U}_l(t) \\ X_1(t)\bar{U}_1(t) & \cdots & X_l(t)\bar{U}_l(t) \end{pmatrix} \tag{A.2}$$

satisfies the linear difference equation (3.1). In fact, using (A.1), we obtain

$$\bar{U}_i(t+1) = [B_{11}(t) + B_{12}(t)X_i(t)]\bar{U}_i(t),$$
$$i = 1, \dots, l, \quad t \in [t_0, t_1 - 1].$$

Using the expression for $X_i(t+1)$ given by (1.1), and the above for $U_i(t+1)$, we easily verify that each block column of (A.2) satisfies (3.1).

Now, we can set a correspondence between the time varying subspace spanned by (A.2) and a solution of (1.1). By (3.3), we can write this solution as

$$X(t) \doteq (X_1(t)\bar{U}_1(t), \dots, X_l(t)\bar{U}_l(t))(\bar{U}_1(t), \dots, \bar{U}_l(t))^{-1}.$$

Now write this as

$$X(t) = \sum_{i=1}^{l} Xi(t)\Pi_i(t),$$

where $\Pi_i(t) \doteq (0 \ \cdots \ 0 \ \bar{U}_i(t) \ 0 \ \cdots \ 0)(\bar{U}_1(t) \ \bar{U}_2(t) \ \cdots \ \bar{U}_l(t))^{-1}$, such that in the first matrix all the entries are zero except for the columns from $n_1 + n_2 + \cdots + n_{i-1} + 1$ to $n_1 + n_2 + \cdots + n_i$, which are equal to $\bar{U}_i(t)$. A straightforward verification shows that $\Pi_i(t)$ is indeed the projection onto $M_i(t)$ along $M_1(t) \oplus \cdots \oplus M_{i-1}(t) \oplus M_{i+1}(t) \oplus \cdots \oplus M_l(t)$, and this completes the proof. ■

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